1. (12 pts) (a) Find the eigenvalues of the matrix \[
\begin{pmatrix}
1 & 2 & 0 \\
0 & 3 & 0 \\
0 & 0 & 4
\end{pmatrix}.
\]
   (b) Find an eigenvector for each eigenvalue.

2. (12 pts) Let \( W \) be the subspace of \( \mathbb{R}^5 \) spanned by \[
\begin{pmatrix}
1 \\
2 \\
0 \\
0 \\
2
\end{pmatrix},
\begin{pmatrix}
2 \\
-1 \\
0 \\
0 \\
-2
\end{pmatrix}, \text{ and } \begin{pmatrix}
0 \\
2 \\
2 \\
1
\end{pmatrix}.
\]
   (a) Find an orthonormal basis for \( W \).
   (b) Write \( (9, 9, 9, 9)^T \) as a sum of a vector in \( W \) and a vector in \( W^\perp \).

3. (6 pts) \( L((a, b, c)^T) = ax(x - 1) + bx + c \) defines a linear transformation \( L \) from \( \mathbb{R}^3 \) to \( P_3 \). Find a matrix for \( L \) using the standard basis for \( \mathbb{R}^3 \) and the basis \( 1, x, x^2 \) for \( P_3 \).

4. (12 pts) A matrix \( A \in \mathbb{R}^{4 \times 4} \) has eigenvalues 1, \(-1\), 2 and 3. What are the eigenvalues and determinants of the following matrices?

   (i) \( A^{-1} \)  (ii) \( A^T \)  (iii) \( A^2 - A \).

5. (10 pts) Let \( \mathbf{v} \) and \( \mathbf{w} \) be nonzero vectors in \( \mathbb{R}^n \). Define \( A \in \mathbb{R}^{n \times n} \) by \( A = \mathbf{v}\mathbf{w}^T \). Prove that \( \mathbf{v} \) is a basis for the column space of \( A \).

   \text{Hint: What is column } k \text{ of } A \text{ in terms of } \mathbf{v} \text{ and } \mathbf{w} ?

\text{THERE ARE MORE PROBLEMS}
6. (10 pts) Given three matrices $A, B, C \in \mathbb{R}^{3 \times 4}$ a student was told to compute a basis for the row space of each matrix and a basis for the null space of each matrix. The following answers were turned in.

<table>
<thead>
<tr>
<th>Row Space</th>
<th>Null Space</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>matrix $A$:</strong></td>
<td>(1, 0, 0, 0)</td>
</tr>
<tr>
<td></td>
<td>(0, 1, -1, 0)</td>
</tr>
<tr>
<td></td>
<td>(0)</td>
</tr>
<tr>
<td></td>
<td>(1, 0, 0, 0)</td>
</tr>
<tr>
<td><strong>matrix $B$:</strong></td>
<td>(0, 1, -1, 0)</td>
</tr>
<tr>
<td></td>
<td>(0, 0, 1)</td>
</tr>
<tr>
<td><strong>matrix $C$:</strong></td>
<td>(1, 0, 0, 0)</td>
</tr>
<tr>
<td></td>
<td>(0, 1, -1, 0)</td>
</tr>
</tbody>
</table>

The grader marked two answers wrong and one answer correct. It is possible to tell which answers are wrong without even knowing the matrices $A, B, C$!

**Question:** Which two answers must be wrong and why?

7. (12 pts) Suppose $A \in \mathbb{R}^{n \times n}$ is nonsingular. For $x, y \in \mathbb{R}^n$, define $\langle x, y \rangle = x^T A^T A y$. Prove that this makes $\mathbb{R}^n$ into an inner product space. That is, verify the three conditions in the definition of an inner product space:

(i) $\langle x, x \rangle \geq 0$ with equality if and only if $x = 0$.

(ii) $\langle x, y \rangle = \langle y, x \rangle$ for all $x$ and $y$ in $\mathbb{R}^n$.

(iii) $\langle \alpha x + \beta y, z \rangle = \alpha \langle x, z \rangle + \beta \langle y, z \rangle$ for all $x, y$ and $z$ in $\mathbb{R}^n$ and all scalars $\alpha$ and $\beta$.

8. (6 pts) Let $A = (1, 0)$. It is easily seen that

$$\det(A^T A) = \det \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = 0 \quad \text{and} \quad \det(AA^T) = \det(1) = 1.$$ 

What is wrong with the following proof that $0 = 1$?

$$0 = \det(A^T A) = \det(A^T) \det(A) = \det(A) \det(A^T) = \det(AA^T) = 1.$$