1. • If \(|x - 2| > 1\), the series diverges since the terms do not go to zero.
  • If \(|x - 2| < 1\), the series converges absolutely by ratio test, or the root test or by comparison with the geometric series \(\sum |x - 2|^n\).
  • If \(x - 2 = +1\), the series is a divergent \(p\)-series.
  • If \(x - 2 = -1\), the series is alternating and converges since the terms decrease to zero.

Putting all this together:

conditional if and only if \(x = 1\),

absolute if and only if \(|x - 2| < 1\) that is, \(1 < x < 3\).

2. (a) The equation is linear: \(y' + \frac{1}{x+1}y = \frac{2x}{x+1}\). You can use the formula for a linear equation or do it from “scratch.” I’ll do the latter. The integrating factor is \(x+1\) and so \((x+1)y' = 2x\). Thus \((x+1)y = x^2 + C\). Setting \(x = 0\) and \(y = 2\) gives \(C = 2\). Thus \(y = \frac{x^2 + 2}{x + 1}\).

Alternatively, the equation written as \((-2x + y)dx + (x + 1)dy = 0\) is exact and one finds \(-x^2 + xy + y = C\). The initial condition \((x, y) = (0, 2)\) gives \(C = 2\).

(b) This is an Euler equation. Even if you don’t recognize it as such, you should recognize it as having a regular singular point and use the same approach. We try a solution of the form \(y = x^r\). This leads to the indicial equation \(r(r-1) + 3r - 3 = 0\) which has solutions \(r = -3, 1\). The general solution is \(y = c_1 x^{-3} + c_2 x\). From the initial conditions, \(c_1 + c_2 = 4\) and \(-3c_1 + c_2 = 0\). Solving: \(c_1 = 1\) and \(c_2 = 3\). Thus we have \(y = x^{-3} + 3x\).

(c) The homogeneous equation is \(y'' - 4y = 0\). The characteristic equation is \(r^2 - 4 = 0\) and so two independent solutions are \(y_1 = e^{2t}\) and \(y_2 = e^{-2t}\). Now we use variation of parameters. The Wronskian is \(W(y_1, y_2) = -4\). By the formula for variation of parameters, a particular solution is

\[
Y(t) = e^{2t} \int 4e^{-2t} \ln t \, dt - e^{-2t} \int 4e^{2t} \ln t \, dt.
\]

The general solution is \(Y(t) + c_1 e^{2t} + c_2 e^{-2t}\).

3. The equilibrium points occur when \(y = n\pi\) for some integer \(n\). An equilibrium point of \(y' = f(y)\) is stable when \(f' < 0\) and unstable when \(f' > 0\), so the unstable points are \(y = 2n\pi\) and the stable ones are \(y = (2n + 1)\pi\). You were asked to find one of each.

4. Setting \(y(x) = \sum a_n x^n\), we obtain

\[
\sum_{n=0}^{\infty} (n + 2)(n + 1)a_{n+2}x^n - \sum_{n=0}^{\infty} na_n x^n - \sum_{n=0}^{\infty} 2a_n x^n = 0,
\]

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which gives the recurrence relation $a_{n+2} = a_n/(n+1)$ for $n \geq 0$. From the initial conditions, $a_0 = 1$ and $a_1 = 0$. You can work out the required nonzero terms. I’ll get the general term. Since $a_1 = 0$, the recurrence gives $a_n = 0$ for all odd $n$. For $n = 2k$,

$$a_{2k} = \frac{a_{2k-2}}{2k-1} = \frac{a_{2k-4}}{(2k-1)(2k-3)} = \cdots = \frac{1}{(2k-1)(2k-3) \cdots 3 \cdot 1}.$$ 

5. Let $y(t) = tv(t)$. Then $y' = tv' + v$ and $y'' = tv'' + 2v'$. Substitution and a bit of calculation gives

$$t^4v'' + (2t^3 - t^2)v' = 0 \quad \text{whence} \quad \frac{dv'}{v'} = \frac{-2t^3 + t^2}{t^4} dt.$$ 

After integration, $\ln |v'| = -2 \ln t - 1/t + C$. Exponentiating and choosing a convenient value for $C$:

$$v' = t^{-2}e^{-1/t} \quad \text{whence} \quad v = e^{-1/t},$$

by, for example, the substitution $-1/t = w$ to get $dw = dt/t^2$. Thus the general solution is $c_1t + c_2te^{-1/t}$.

6. Since $x^2(1-x^2) = 0$ if and only if $x = 0, \pm 1$, these are the singular points. The point $x = 0$ is irregular because $xQ(x)/P(x)$ does not have a power series at $x = 0$. The other two are regular; e.g., for $x = 1$, both $(x-1)Q(x)/P(x)$ and $(x-1)^2R(x)/P(x)$ have power series at $x = 1$.

7. Let $Y(s)$ be the Laplace transform of $y(t)$. From the table,

$$\mathcal{L}\{y'\} = sY - f(0) = sY - 1 \quad \text{and} \quad \mathcal{L}\{y''\} = s^2Y - sf'(0) - sf(0) = s^2Y - s.$$ 

What about $\mathcal{L}\{g(t)\}$? Many people forgot the definition of the Laplace transform:

$$\mathcal{L}\{g(t)\} = \int_0^\infty e^{-st}g(t) \, dt.$$ 

Hence

$$\mathcal{L}\{g(t)\} = \int_0^2 e^{-st} \, dt + \int_2^\infty 0 \, dt = -e^{-st}/s \bigg|_{t=2}^{t=\infty} = \frac{1 - e^{-2s}}{s}.$$ 

Hence

$$s^2Y - s - 2(sY - 1) + Y = \frac{1 - e^{-2s}}{s}.$$ 

Solving for $Y$, we get

$$Y = \frac{s - 2 + (1 - e^{-2s})/s}{s^2 - 2s + 1} = \frac{(s - 1)^2 - e^{-2s}}{s(s - 1)^2} = \frac{1}{s} - \frac{1}{s(s - 1)^2e^{2s}},$$

or any of a variety of equivalent forms.