1. In the standard notation, \( p(x) = 1/(1-x)(1+x)^2 \) and \( q(x) = 1/(1+x)(1-x)^2 \).
   (a) Since \( p(x) \) and \( q(x) \) are okay except at \( x = \pm 1 \), these are the singular points. Regularity requires that \( (x-x_0)p(x) \) and \( (x-x_0)^2q(x) \) have power series at \( x_0 \). When \( x_0 = 1 \), they both have power series, so this point is regular. When \( x_0 = -1 \), \( p(x) \) does not have a power series so this point is irregular (or you can say “not regular”).
   (b) It converges: The power series for \( p \) and \( q \) about \( x_0 = 0 \) converge for \( |x| < 1 \). By a theorem in the book, so does the power series for \( y(x) \).

2. We have \( (s^2Y(s) - s - 2) + (Y(s) - 1) = 1/s + 2/(s-1) \). Thus
   \[
   Y(s) = \frac{1/s + 2/(s-1) + s + 3}{s^2 + 1} = \frac{s^3 + 2s^2 - 1}{s(s-1)(s^2 + 1)}.
   \]

3. Since the water level would rise 100 feet in 5 days, it is rising at the rate of 20 feet per day. This is the rate at which water is flowing in. Since it is flowing out at the rate of \( 5h^{1/2} \) feet per day, the differential equation is \( h' = 20 - 5h^{1/2} \). The initial condition is \( h(0) = 0 \).

4. (a) The equation is linear: \( xy' + 2y = 3x \). The integrating factor is \( x \), so \( (x^2y)' = 3x^2 \). Thus \( x^2y = x^3 + C \). Since \( y(1) = 2 \), \( 1^2 \times 2 = 1^3 + C \) and so \( C = 1 \). Hence \( y = x + x^{-2} \).
   Alternatively, the equation is homogeneous
   (b) The equation is separable, so \( \ln |y| = 3x - 2 \ln |x| + C \). Since \( y(1) = 2 \), \( x \ y \) are positive and \( \ln 2 = 3 + C \). Hence \( C = \ln 2 - 3 \). One can exponentiate to get a nicer form: \( y = 2e^{3x-3}/x^2 \).
   Alternatively, the equation is linear.
   (c) By undetermined coefficients, variation of parameters, or observation, \( y = -t \) is a particular solution. The homogeneous equation \( y'' - y = 0 \) has characteristic equation \( r^2 - 1 = 0 \) and so the general solution is \( y = c_1e^t + c_2e^{-t} - t \). Using the initial conditions: \( c_1 + c_2 = 0 \) and \( c_1 - c_2 - 1 = 0 \). Thus \( c_1 = 1/2 \) and \( c_2 = -1/2 \). The equation can also be solved by Laplace transforms: \( s^2Y - Y = 1/s^2 \). By algebra and partial fractions,
   \[
   Y = \frac{1}{s^2(s^2 - 1)} = \frac{1/2}{s - 1} - \frac{1/2}{s + 1} - \frac{1}{s^2}.
   \]
   (d) The equation is homogeneous. Set \( y = xv \) and \( y' = xv' + v \) to obtain \( xv' + v = 1 - v + v^2 \). Thus \( xv' = (1 - v)^2 \). Separate variables and integrate to get \( (1 - v)^{-1} = \ln x + C \). Thus \( (1 - y/x)^{-1} = \ln x + C \). From the initial condition, \( C = 1 \). You can solve for \( y \) if you wish: \( y = x - x/\ln(ex) \).
5. We set $y_2 = y_1 v = xv$. Since $y_2' = v + xv'$ and $y_2'' = 2v' + xv''$,
\[ 0 = x^3y_2'' + xy_2' - y_2 = (2x^3v' + x^4v'') + (xv + x^2v') - xv = x^4v'' + (2x^3 + x^2)v'. \]

Separating variables:
\[ \frac{dv'}{v'} = \frac{-(2x + 1)dx}{x^2}. \]

Thus a particular solution is $\ln v' = -2 \ln x + 1/x^2$. Exponentiating: $v' = x^{-2}e^{1/x}$. Integrating: $v = -e^{1/x}$. This gives $y_2 = -xe^{1/x}$ as an independent solution. (Since we can multiply by a constant, any solution of the form $y_2 = c_1xe^{1/x} + c_2x$ is acceptable if $c_1 \neq 0$.

6. We set $y = \sum a_n x^n$, differentiate it twice, substitute into the equation, and look at the coefficient of $x^n$ to obtain the recursion
\[ (n + 2)(n + 1)a_{n+2} - n(n - 1)a_n + 4(n + 1)a_{n+1} + 6a_n = 0. \]

Thus
\[ a_0 = 1, \quad a_1 = -3, \quad \text{and} \quad a_{n+2} = \frac{(n^2 - n - 6)a_n - 4(n + 1)a_{n+1}}{(n + 2)(n + 1)}. \]

With $n = 0, a_2 = 3$. With $n = 1, a_3 = -1$. With $n = 2, a_4 = 0$. With $n = 3, a_5 = 0$. From now on $a_{n+2} = 0$ since it depends only on the previous two values $a_n$ and $a_{n+1}$.

7. (a) This is an Euler equation, so $y = x^r$ where $2r(r - 1) + 3r - 1 = 0$. Thus $r = -1$ and $r = 1/2$ and so the general solution is $y = c_1/x + c_2x^{1/2}$.

(b) Any method that produces a particular solution is acceptable. This includes using undetermined coefficients even though there is no reason that method should give a solution since undetermined coefficients is for constant coefficient linear equations.

The simplest approach is to note that, in solving Euler’s equation, the left side is a function of $r$ times $x^r$. Since we want to end up with $x^2$, we try $y = Cx^2$ for a particular solution. Then we get $4Cx^2 + 6Cx^2 - Cx^2 = 9x^2$. Hence $C = 1$ and so the general solution is $y = c_1/x + c_2x^{1/2} + x^2$.

Alternatively, we can use the formula for variation of parameters (p. 176). Note that we must divide the given equation by $2x^2$ so that the coefficient of $y''$ is one. Thus $g(x) = 9/2$. After some calculations, $W(y_1, y_2) = 3/2(x^{3/2})$ and $Y = x^2$.

Alternatively, you can use the trick I mentioned for converting an Euler equation to a constant coefficient equation: Set $\ln x = t$. The given equation becomes $2d^2y/dt^2 + dy/dt - y = 9e^{2t}$. This can be solved in various ways. The general solution is $y = c_1e^{-t} + c_2e^{t/2} + e^{2t}$. Replace $t$ with $\ln x$ to get the solution.