1. (a) The graph is a triangle $a, b, c$ with two additional edges attached at $a$.

(b) There are three spanning trees. Each tree is obtained by removing exactly one of the edges $\{a, b\}, \{a, c\}, \{b, c\}$.

(c) The tree obtained by removing $\{b, c\}$ is not lineal. The other two trees are.

(d) This can be done in various ways. Here’s one. Color $a$ ($x$ ways), then color $c$, $d$ and $e$ ($x - 1$ ways each since the only constraint is that they differ from the color of $a$). Finally, color $b$ ($x - 2$ ways since it must differ from both $a$ and $c$). The answer is $x(x - 1)^3(x - 2)$.

2. We have $G_0 = G_1 = x$ and $G_{11} = x^2$. Thus $G_{0*} = \frac{1}{1 - x}$ and $G_{(11)*} = \frac{1}{1 - x^2}$. Hence

$$G_{00*} = \frac{x}{1 - x}, \quad G_{00*(11)*1} = \frac{x}{1 - x} \frac{x}{1 - x^2}, \quad G_{(00*)(11)*1} = \frac{1}{1 - x} \frac{x}{1 - x^2}$$

and so

$$A(x) = \frac{1}{1 - x} - \frac{x}{1 - x^2} \frac{x}{1 - x} = \frac{(1 - x^2)x}{(1 - x)(1 - x^2) - x^2} = \frac{(1 - x^2)x}{1 - x - 2x^2 + x^3}.$$

Thus $P(x) = x(1 - x^2)$.

3. Multiply both sides of (1) by the denominator $1 - x - 2x^2 + x^3$ and find the coefficient of $x^n$ on both sides. Since $P(x)$ is a cubic, we have

$$a_n - a_{n-1} - 2a_{n-2} + a_{n-3} = 0$$

for $n > 3$. Rearranging gives the recursion $a_n = a_{n-1} + 2a_{n-2} - a_{n-3}$.

4. This can be done using Principle 11.6 or Example 11.27. The singularity closest to the origin is the smallest root of the denominator of $A(x)$, namely $\beta$. We have $A(x) = (1 - x/\beta)^{-1}g(x)$ where

$$g(x) = \frac{P(x)}{-\beta(x - \alpha)(x - \gamma)} \quad \text{and} \quad g(\beta) = \frac{P(\beta)}{\beta(\beta - \alpha)(\gamma - \beta)}.$$

Thus $A = g(\beta), B = 0$ and $C = 1/\beta$.

5. This type of problem was discussed in Example 10.9 and in class. The answer is

$$\left[ x^n \right] A_y(x, 1)$$

$$\left[ x^n \right] A(x, 1).$$