1. (28 pts.) Consider the simple graph $G = (V, E)$ with $V = \{a, b, c, d, e\}$ and $E = \{\{a, b\}, \{a, c\}, \{b, c\}, \{a, d\}, \{a, e\}\}$.
   
   (a) (6 pts.) Sketch the graph $G$.
   
   (b) (6 pts.) Give all spanning trees of $G$.
   
   (c) (6 pts.) Think of each spanning tree in (b) as rooted at the vertex $a$. For each of these rooted trees, indicate whether or not it is a lineal spanning tree of $G$.
   
   (A lineal spanning tree is also called a depth first spanning tree.)
   
   (d) (10 pts.) Compute the chromatic polynomial of $G$.

The sequences of zeroes and ones that begin and end with zero such that all maximal strings of ones are of odd length are described by the regular expression

$$
(00^* (11)^* 1)^* 00^*.
$$

You do not need to prove this. If $a_n$ is the number of such $n$-long sequences, then the generating function $A(x) = \sum a_n x^n$ has the form

$$
\frac{P(x)}{1 - x - 2x^2 + x^3}
$$

for some third degree polynomial $P(x)$.

**Each of the following problems can be done independently of the others.**

2. (10 pts.) Using (1), derive the formula for $A(x)$, including a formula for $P(x)$.

3. (8 pts.) Find $k$ and constants $c_1, c_2, \ldots, c_k$ so that $a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k}$ for all sufficiently large $n$. You need not find the initial conditions.

4. (8 pts.) It turns out that

$$
 x^3 - 2x^2 - x + 1 = (x - \alpha)(x - \beta)(x - \gamma)
$$

where $\alpha = -0.801937\ldots$, $\beta = 0.554958\ldots$ and $\gamma = 2.246979\ldots$. Find constants $A$, $B$ and $C$ so that $a_n \sim A n^B C^n$. You may express $A$, $B$ and $C$ in terms of $\alpha$, $\beta$, $\gamma$ and $P$, so there is no need for a calculator.

5. (8 pts.) Let $A(x, y)$ be the generating function for the sequences counted by (1), where the coefficient of $x^n y^k$ is the number of $n$-long sequences with exactly $k$ ones. You do not need to compute this generating function.

Write down a formula for the average number of ones in an $n$-long sequence in terms of the coefficients of $A(x, y)$ and related generating functions. An example (but **WRONG**) of such an expression is $\left( [x^n y^k] \left( A(x, y) \right)^2 \right) / \left( [x^n] A_x(x, 1) \right)^2$, where $A_x = \partial A / \partial x$.