1. (45 pts.) (a) Compute the rank of the permutation 1, 5, 3, 4, 2 when the permutations of \{1, 2, 3, 4, 5\} are listed in direct insertion order.
(b) What permutation immediately follows 1, 5, 3, 4, 2 in direct insertion order?
(c) What permutation immediately follows 1, 5, 3, 4, 2 in lex order?

2. (20 pts.) Find the unlabeled rooted plane tree with 7 leaves and rank 60.

\[ b_1 = b_2 = 1, \quad b_3 = 2, \quad b_4 = 5, \quad b_5 = 14, \quad b_6 = 42, \quad b_7 = 132. \]

3. (30 pts.) Let \( C_n(k) \) be the number of times position \( k \) is changed in the Gray code for \( n \)-long vectors of zeroes and ones given in the text. You may use the following fact without proving it:

\[
C_n(k) = \begin{cases} 
1 & \text{if } k = 1, \\
2C_{n-1}(k-1) & \text{if } 1 < k \leq n, \\
0 & \text{if } k > n.
\end{cases}
\]

(a) Tabulate values of \( C_n(k) \) for \( 1 \leq k \leq n \leq 4 \).
(b) State and prove a simple formula for \( C_n(k) \) that does not involve a recursion.

4. (30 pts.) Let \( b_n \) be the number of binary unlabeled rooted plane trees with \( n \) leaves. It is known that \( b_n < 4b_{n-1} \) for \( n > 1 \) and you may use this fact without proof. (It can be proved by using Exercise 9.1.12.)

(a) Show that, for more than \( b_n/4 \) of these trees with \( n \) leaves, the left subtree consists of just a single leaf when \( n > 1 \).
In other words, show that if the two edges leading from the root go to trees \( T_1 \) and \( T_2 \), then we have \( |T_1| = 1 \) in more than \( b_n/4 \) of the cases.

**Hint:** What does the term \( b_ib_{n-i} \) in \( b_n = b_1b_{n-1} + b_2b_{n-2} + \cdots + b_{n-1}b_1 \) count?

(b) Find a constant \( P > 1/4 \) such that the following statement is true for \( n > 1 \) about those binary trees with \( n \) leaves.

“In more than \( Pb_n \) of them, the left subtree contains at most 2 leaves.”

**To receive credit, you must prove that your value for \( P \) works.**