1. Let $G = (V, E)$ where $V = \{0, 1, a, b, A, B\}$ and 
$$E = \{(0, 1), (0, a), (0, b), (0, A), (0, B), (a, b), (A, B)\}.$$ 

**Sketch** the simple graph $G$ and **compute** its chromatic polynomial. 

**A.** We omit the sketch. There are various ways to compute the chromatic polynomial. One can use formulas, but the easiest is to appeal directly to the definition. Given $x$ colors, there are $x$ ways to color vertex 0. There are then $x - 1$ ways to color each of 1, $a$, and $A$ since they must differ from 0. There are then $x - 2$ ways to color each of $b$ and $B$. (For example, $b$ must differ from 0 and $a$, which leaves $x - 2$ colors for it.) By the Rule of Product from Chapter 1, $P_G(x) = x(x - 1)^3(x - 2)^2$. 

2. **Compute** the rank of the binary RP-tree shown here. 

For your information, $b_1 = b_2 = 1$, $b_3 = 2$, $b_4 = 5$, $b_5 = 14$, $b_6 = 42$, and $b_7 = 132$. 

**A.** Let $B$ be the given tree and let $T = \begin{array}{c} \end{array}$. Then 

$$\text{RANK}(B) = b_1b_5 + b_2b_4 + \text{RANK}(T)b_3 + \text{RANK}(T).$$ 

You can compute $\text{RANK}(T)$ or note that $T$ is the second of the two trees counted by $b_3$ and so has rank 1. Thus the answer is $14 + 5 + 2 + 1 = 22$. 

3. The local description of a decision tree for constructing sequences of A’s and B’s is given below. The notation $BA S(n - 2)$ means place $BA$ in front of each sequence produced by $S(n - 2)$. 

$$S(1) \quad S(2) \quad S(n) \quad (n \geq 3)$$ 

$$\begin{array}{c} A \quad B \quad A \quad S(1) \quad BA \quad A \quad S(n - 1) \quad BA \quad S(n - 2) \end{array}$$ 

Let $S^*(n)$ denote the entire decision tree. Thus $S^*(1) = S(1)$ and $S^*(2)$ has the three leaves $AA$, $AB$, and $BA$. 

(a) **Find** a recursion for $s_n$, the number of leaves of $S^*(n)$. 

*Remember to include initial conditions.* 

**A.** From the pictures, we read off $s_1 = 2$, $s_2 = s_1 + 1 (= 3$, if you wish), and $s_n = s_{n-1} + s_{n-2}$ for $n \geq 3$. 

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(b) **Prove** that the leaves of $S^*(n)$ are sequences of length $n$ and that their order from left to right is alphabetic.

**A.** We prove it by induction on $n$. For $n = 1$, it is clear from the picture. For $n = 2$, the entire tree has leaves AA, AB, and BA from left to right. Now we use induction for $n \geq 3$. Since A precedes B, the leaves of $A S^*(n - 1)$ precede the leaves of $BA S^*(n - 2)$ in alphabetic order. The leaves of $S^*(n - 1)$ all have length $n - 1$ and are in alphabetic order. Thus the leaves of $A S^*(n - 1)$ all have length $n$ and are in alphabetic order. The leaves of $S^*(n - 2)$ all have length $n - 2$ and are in alphabetic order. Thus the leaves of $BA S^*(n - 2)$ all have length $n$ and are in alphabetic order. This completes the proof.

4. A binary RP-tree has information stored at each leaf vertex. Each non-leaf vertex may or may not have information stored at it. Let $t_n$ be the number of such trees with information stored at exactly $n$ vertices and let $T(x) = \sum t_n x^n$ be the generating function. The following picture shows some of the nine trees that contribute to $t_4$. An empty circle indicates a vertex with no information.

Find a formula for $T(x)$ similar to the formula $B(x) = x + B(x)^2$ we found for binary RP-trees. To receive credit you must justify your formula; that is, explain how you got it.

**A.** A tree of the desired type is either
(a) a single vertex with information (since it is a leaf and so has information) OR
(b) a root with information joined to two trees of the same type OR
(c) a root without information joined to two trees of the same type.

The generating function for a vertex with information is $x^1 = x$ and that for a vertex without information is $x^0 = 1$. Using the Rule of Product in (b) and (c) and the Rule of Sum to combine the results we have

$$T(x) = x + xT(x)^2 + 1T(x)^2 = x + (x + 1)T(x)^2.$$