1. In each case, give an example or explain why none exists.
   (a) A permutation \( f \) of \( \{1, 2, 3, 4, 5\} \) such that, for some \( x \in \{1, 2, 3, 4, 5\} \), \( f^{20}(x) \neq x \).
   (b) A permutation \( f \) of \( \{1, 2, 3, 4, 5\} \) such that, for every \( x \in \{1, 2, 3, 4, 5\} \), \( f^{20}(x) \neq x \).
   (c) A tree with exactly 10 vertices and exactly 10 edges.

2. In each case, give an example or explain why none exists.
   (a) A function \( f(n) \) such that \( f(n) \) is \( O(n^2) \) but \( f(n) \) is not \( \Theta(n^2) \).
   (b) A function \( f(n) \) such that \( f(n) \) is \( O(n \log n) \) but \( f(n) \) is not \( O(n^2) \).
   (c) A probability space \((U, P)\) and two subsets \( S \) and \( T \) of \( U \) such that \( P(S) = P(T) = 2/3 \) and \( S \neq T \).

3. A fair die is tossed. If \( n \) is the value that is seen, define the random variable \( X \) by \( X = |n - 3| \)
   (a) Compute the probability that \( X = k \) for \( k = 0, 1, 2, 3, 4, 5, 6 \).
   (b) Compute the mean and variance of \( X \). \textit{Do the arithmetic.}

4. The platoon commander knows:
   - If the air strike is successful, there is a 60% probability that the ground forces \textit{will not} encounter enemy fire.
   - If the air strike is not successful, there is a 80% probability that the ground forces \textit{will} encounter enemy fire.
   - There is a 70% probability that the air strike will be successful.

   Answer the following questions.
   (a) **What is the probability** that the ground forces \textit{will not} encounter enemy fire?
   (b) The ground forces did not encounter enemy fire. **What is the probability** that the air strike was successful?

5. After being dealt 4 cards, I have 3 of a kind and a 4th card that has a different face value.
   (a) **How many** such hands of 4 cards are there? (For counting, the order cards are dealt does not matter, only what is in the hand.)
   (b) I will be dealt a 5th card. **What is the probability** that, given the 4 cards I already have, I will end up with a hand that contains either 4 of a kind or a full house?
      (A full house is a pair and 3 of a kind.)

THERE IS MORE
6. **Prove**: If a graph has \( v \) vertices and \( n \) connected components, then it has at least \( v - n \) edges.

**Hint**: A tree with \( t \) vertices has \( t - 1 \) edges.

7. Define \( a_n \) by \( a_0 = 1 \) and the recursion \( a_n = (n/a_{n-1}) + a_{n-1} \) for \( n > 0 \).

**Guess and prove** a formula for \( a_n \).

**Suggestion**: To help with your guessing, compute the first few values of \( a_n \).

8. The following algorithm computes \( x^n \) for \( n \) a nonnegative integer, where \( x \) is a complicated object and \( \text{MULT} \) is a procedure that multiplies such objects.

```
POW(x,n)
If (n=0) Return 1
Else
  Let q and r be the quotient and remainder when n is divided by 2.
  // Thus q = n/2 rounded down and r = n - 2q, which is 0 or 1.
  y = MULT(x,x)
  z = POW(y,q) // Remark: A recursive call.
  If (r=0) Return z
  Else Return MULT(x,z)
End if
End if
End
```

**Find** a function \( T(n) \) so that the number calls of \( \text{MULT} \) is \( \Theta(T(n)) \).

**Hint**: Use the Master Theorem for Recursions.

**Theorem (Master Theorem for Recursions)** Suppose that there are

(i) numbers \( N \) and \( 0 < c < 1 \),
(ii) a sequence \( a_1, a_2, \ldots \),
(iii) functions \( s_1, s_2, \ldots, s_w \), and \( T \)

such that

(a) \( T(n) > 0 \) for all \( n > N \) and \( a_n \geq 0 \) for all \( n > N \);
(b) \( T(n) = a_n + T(s_1(n)) + T(s_2(n)) + \cdots + T(s_w(n)) \) for all \( n > N \);
(c) \( a_n \) is \( \Theta(n^b) \) for some \( b \geq 0 \);
(d) \( |s_i(n) - cn| \) is \( O(1) \) for \( i = 1, 2, \ldots, w \).

Let \( d = -\log(w)/\log(c) \). Then

\[
T(n) \text{ is } \begin{cases} \Theta(n^d) & \text{if } b < d, \\
\Theta(n^d \log n) & \text{if } b = d, \\
\Theta(n^b) & \text{if } b > d. 
\end{cases}
\]