Calculating $N_{j,4}(u_i)$. — Sam Buss — February 11, 2004

The following will give formulas for calculating the values α_i, β_i, and γ_i for the computation of interpolating B-spline curves of order 4.

Let the knot vector be $[u_0, u_1, u_2, \ldots, u_\ell]$. These knots are not necessarily unique. Note they are renamed from the conventions in the last section of Chapter VIII.

The values $N_{i-1,4}(u_i)$, $N_{i-2,4}(u_i)$ and $N_{i-3,4}(u_i)$ can be calculated using the the de Boor algorithm as follows.

$\underline{N_{i-1,4}(u_i)}$: Let $\mathbf{p}_{i-1} = 1$ and other $\mathbf{p}_j$'s equal zero.

$$\begin{aligned}
\mathbf{p}_{i-1}^{(1)} &= Lerp\left(\mathbf{p}_{i-2}, \mathbf{p}_{i-1}, \frac{u-u_{i-1}}{u_{i+2}-u_{i-1}}\right) = Lerp\left(0, 1, \frac{u-u_{i-1}}{u_{i+2}-u_{i-1}}\right) = \frac{u-u_{i-1}}{u_{i+2}-u_{i-1}} \\
\mathbf{p}_{i-2}^{(1)} &= Lerp\left(\mathbf{p}_{i-3}, \mathbf{p}_{i-2}, \frac{u-u_{i-2}}{u_{i+1}-u_{i-2}}\right) = Lerp\left(0, 0, \frac{u-u_{i-2}}{u_{i+1}-u_{i-2}}\right) = 0 \\
\mathbf{p}_{i-1}^{(2)} &= Lerp\left(0, \frac{u-u_{i-1}}{u_{i+2}-u_{i-1}}, \frac{u-u_{i-1}}{u_{i+1}-u_{i-1}}\right) = \frac{(u-u_{i-1})^2}{(u_{i+1}-u_{i-1})(u_{i+2}-u_{i-1})}.
\end{aligned}$$

This last value, with u replaced by u_i, is $N_{i-1,4}(u_i)$.

$\underline{N_{i-2,4}(u_i)}$: Let $\mathbf{p}_{i-2} = 1$ and other $\mathbf{p}_j$'s equal zero.

$$\begin{aligned}
\mathbf{p}_{i-1}^{(1)} &= Lerp\left(1, 0, , \frac{u-u_{i-1}}{u_{i+2}-u_{i-1}}\right) = \frac{u_{i+2}-u}{u_{i+2}-u_{i-1}} \\
\mathbf{p}_{i-2}^{(1)} &= Lerp(\left(0, 1, \frac{u-u_{i-2}}{u_{i+1}-u_{i-2}}\right) = \frac{u-u_{i-2}}{u_{i+1}-u_{i-2}} \\
\mathbf{p}_{i-1}^{(2)} &= Lerp\left(\frac{u-u_{i-2}}{u_{i+1}-u_{i-2}}, \frac{u_{i+2}-u}{u_{i+2}-u_{i-1}}, \frac{u-u_{i-1}}{u_{i+1}-u_{i-1}}\right) \\
&= \frac{u_{i+1}-u}{u_{i+1}-u_{i-1}} \cdot \frac{u-u_{i-2}}{u_{i+1}-u_{i-2}} + \frac{u-u_{i-1}}{u_{i+1}-u_{i-1}} \cdot \frac{u_{i+2}-u}{u_{i+2}-u_{i-1}}.
\end{aligned}$$

This last value, with u replaced by u_i, is $N_{i-2,4}(u_i)$.

$\underline{N_{i-3,4}(u_i)}$: Let $\mathbf{p}_{i-3} = 1$ and other $\mathbf{p}_j$'s equal zero.

$$\begin{aligned}
\mathbf{p}_{i-1}^{(1)} &= Lerp\left(0, 0, \frac{u-u_{i-1}}{u_{i+2}-u_{i-1}}\right) = 0 \\
\mathbf{p}_{i-2}^{(1)} &= Lerp\left(1, 0, \frac{u-u_{i-2}}{u_{i+1}-u_{i-2}}\right) = \frac{u_{i+1}-u}{u_{i+1}-u_{i-2}} \\
\mathbf{p}_{i-1}^{(2)} &= Lerp\left(\frac{u_{i+1}-u}{u_{i+1}-u_{i-2}}, 0, \frac{u-u_{i-1}}{u_{i+1}-u_{i-1}}\right) = \frac{(u_{i+1}-u)^2}{(u_{i+1}-u_{i-1})(u_{i+1}-u_{i-2})}.
\end{aligned}$$

This last value, with u replaced by u_i, is $N_{i-3,4}(u_i)$.

For example, when defining an interpolating B-spline: if we use the standard knot vector

$$[0,0,0,0,1,2,3,4,5,6,\ldots,n-1,n,n,n,n],$$

then we have

$$\begin{aligned}
\alpha_1 &= N_{1,4}(1) = \frac{(2-1)^2}{(2-0)(2-0)} = \frac{1}{4} \\
\beta_1 &= N_{2,4}(1) = \frac{(2-1)}{(2-0)} \cdot \frac{(1-0)}{(2-0)} + \frac{(1-0)}{(2-0)} \cdot \frac{(3-1)}{(3-0)} = \frac{7}{12} \\
\gamma_1 &= N_{3,4}(1) = \frac{(1-0)^2}{(3-0)(2-0)} = \frac{1}{6}.
\end{aligned}$$

Since $\alpha_i + \beta_i + \gamma_i = 1$, the value β_i can also be calculated by

$$\beta_i = 1 - \alpha_i - \gamma_i.$$

This means that the more complicated formula for $N_{i-2,4}(u_i)$ does not need to be used!